Online

14th Middle European Mathematical Olympiad 2020

Contest Problems with Solutions

The Problem Selection Committee

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gratefully received

63 problem proposals submitted by 5 countries:

Austria — Croatia — Czech Republic — Poland — Slovakia

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I-1

Let $\mathbb{N}$ be the set of positive integers. Determine all positive integers k for which there exist functions $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{N}$ such that g assumes infinitely many values and such that

$$f^{g(n)}(n) = f(n) + k$$

holds for every positive integer n.

(*Remark.* Here, f^i denotes the function f applied i times, i.e., $f^i(j) = \underbrace{f(f(\ldots f(f}_{i \text{ times}}(j))\ldots))$.)

Answer. Such functions exist if and only if $k \geq 2$.

Solution 1. Suppose that $k = 1$ and that f and g satisfy the desired conditions.

Claim. *There exist no n and $m \geq 1$ with $f^m(n) = n$.*

Proof. Suppose that $f^m(n) = n$ for some n and $m \geq 1$. Consider the orbit of n, i.e. the set $A = \{f(n), f^2(n), \ldots, f^{m-1}(n), f^m(n)\}$. Clearly, $f^r(n) \in A$ for all $r \geq 0$. Let $t = f^s(n) = \max A$ and denote $u = f^{s+m-1}(n)$ so that $f(u) = t$. Then $t+1 = f(u)+1 = f^{g(u)}(u) = f^{g(u)+s+m-1}(n) \in A$, contradicting $t = \max A$. □

Claim. $f(n) \geq n+1$ *for all n.*

Proof. Suppose $f(n) \leq n$.

We show inductively that for all $r \geq 0$ there exists $s \geq 1$ with $f^s(n) = f(n) + r$. For $r = 0$ take $s = 1$. Induction step: suppose that $f^s(n) = f(n) + r$ for some $s \geq 1$. Denote $t = f^{s-1}(n)$ and note that $f(n)+r+1 = f^s(n)+1 = f(t)+1 = f^{g(t)}(t) = f^{g(t)+s-1}(n)$, hence $s' = g(t)+s-1 \geq 1$ works for $r+1$.

In particular, setting $r = n - f(n) \geq 0$ we see that $f^s(n) = n$ for some $s \geq 1$. This contradicts the previous claim. □

Now, for all m and n we obtain

$$f^m(n) \geq f^{m-1}(n) + 1 \geq \ldots \geq f(n) + m - 1.$$

Set $m = g(n)$. We obtain $f(n)+1 = f^{g(n)}(n) \geq f(n) + g(n) - 1$, hence $g(n) \leq 2$ for all n. This contradicts the assumption that g is unbounded.

Now, let $k \geq 2$. We construct f and g satisfying desired conditions.

Let $n_1 < n_2 < n_3 < \ldots$ be the sequence consisting of all positive integers not divisible by k (i.e. $n_{i(k-1)+j} = ik + j$ for any $i \geq 0$, $j \in \{1, 2, \ldots, k-1\}$). Consider the sequence

$$k, n_1, 2k, n_2, n_3, n_4, 3k, n_5, \ldots, n_9, 4k, n_{10}, \ldots, n_{16}, 5k, \ldots, ik, n_{(i-1)^2+1}, \ldots, n_{i^2}, (i+1)k, \ldots$$

Note that every positive integer occurs in this sequence exactly once and for every n the number $n+k$ appears after n. For every n let $f(n)$ be the successor of n in this sequence and let $g(n)$ be the number of terms in this sequence between $f(n)$ and $f(n)+k$ (inclusive — we count $f(n)$ and $f(n)+k$ as well). By previous remarks, f and g are well defined and satisfy $f^{g(n)}(n) = f(n)+k$. Moreover, $g(n_{i^2}) = 2i + 3$ for any i, hence g is unbounded.

Solution 2. We will prove that if $k = 1$ then g is necessarily bounded.

The given equation implies that if $m = f(n)$ is in the image of f then $m + 1 = f(n) + 1 = f(f^{g(n)-1}(n))$ is in the image of f as well. Let $f(a)$ be the minimum of the image of f. Then the image of f is equal to $\{f(a), f(a) + 1, \ldots\}$.

However, an easy inductive argument shows that for every m the number $f(a) + m$ is of the form $f^n(a)$. Hence the set $\{f(a), f^2(a), \ldots\}$ is also equal to the image of f.

If $f^x(a) = f^y(a)$ for some $x > y$, then the sequence $a_n = f^n(a)$ is eventually periodic with a period $x - y$, but then the set $\{f(a), f^2(a), \ldots\}$ is finite, which is a contradiction.

Therefore, for every $n \in \mathbb{N}_0$, there exists a unique positive integer x_n such that $f^{x_n}(a) = n+f(a)$, and conversely, for all $x \geq 1$ the number $f^x(a)$ is of the form $m + f(a)$ for some $m \geq 0$. In other words, the map $\mathbb{N}_0 \ni n \mapsto x_n \in \mathbb{N}$ is bijective.

Furthermore, $f^{x_{n+1}}(a) = (f(a)+n)+1 = f^{x_n}(a)+1 = f(f^{x_n-1}(a))+1 = f^{g(f^{x_n-1}(a))}(f^{x_n-1}(a)) = f^{x_n-1+g(f^{x_n-1}(a))}(a)$, which implies $x_{n+1} = x_n - 1 + g(f^{x_n-1}(a)) > x_n$, since $g(t) > 1$ for all t.

Therefore, the map $n \mapsto x_n$ is a strictly increasing bijection from $\mathbb{N}_0$ to $\mathbb{N}$ which gives $x_n = n+1$ for all n. Thus $f^{n+1}(a) = f(a) + n$, which implies $f(f(n)) = f(n) + 1$ for all $n \in \mathbb{N}$, hence $g(n) = 2$ for all $n \in \mathbb{N}$. Obviously, this means g is bounded.

Suppose now that $k \geq 2$. We will give an explicit example of functions f and g satisfying required properties.

For each positive integer n, let $f(k^n) = nk + 1$ and let $f(nk + 1) = k^n + 2$.

For each positive integer a which is not a power of k, let $f(ak) = ak+2$. For any other positive integer x, let $f(x) = x + 1$. Define $g(n)$ to be the smallest m such that $f^m(n) = f(n) + k$. Note that g is well-defined. Now, $g(nk + 1)$ is equal to the number of integers m such that $k^n + 2 \leqslant m \leqslant k^{n+1}$ and k doesn't divide $m-1$. For n large enough, the number of such integers is bigger than n. Therefore, g is also unbounded.

I-2

We call a positive integer N *contagious* if there exist 1000 consecutive non-negative integers such that the sum of all their digits is N. Find all contagious positive integers.

Answer. All $N \geq 13500$.

Any solution naturally splits into two parts:

Part 1: Showing that no $N < 13500$ is contagious.

Part 2: Showing that all $N \geq 13500$ are contagious.

We present one approach to Part 1 and three approaches to Part 2 (by direct construction, by induction, and by discrete continuity).

Part 1. We make the following observation:

(T) Consider a block of 1000 consecutive non-negative integers. Then the last three digits of those numbers (prepended by zeros if needed) form a set $\{000, 001, \ldots, 999\}$.

Thus, given any such block, the sum of the last three digits alone equals $3 \cdot 100 \cdot (0+1+\cdots+9) = 13500$ (since each of the digits $0, 1, \ldots, 9$ occurs 100 times in each of the 3 positions). Therefore no integer less than 13500 is contagious.

Part 2, by direct construction. Fix $N \geq 13500$ and write the "remaining" digit sum as $N - 13500 = d \cdot 1000 + r$, where $d \geq 0$ and $r \in [0, 999]$ are non-negative integers. Write $r = \overline{r_2 r_1 r_0}$ as a 3-digit number (prepended by zeros if needed). Consider a number

$$X = \overline{\underbrace{11\ldots1}_{d \text{ times}} r_2 r_1 r_0}$$

formed by concatenating d copies of the digit 1 and the digits r_2, r_1, r_0. (If $d = 0$ set $X = r$.) We claim that the total digit sum of the 1000 consecutive non-negative integers $X, X+1, \ldots, X+999$ equals N. Note that:

(a) Ignoring the last three digits, the $1000 - r$ numbers $X, \ldots, X + (999 - r)$ have digit sum $d \cdot 1 = d$ each and the next r numbers $X + (1000 - r), \ldots, X + 999$ have digit sum $(d-1) \cdot 1 + 2 = d + 1$ each.

(b) As in Part 1, the last three digits of all the 1000 numbers add up to 13500.

Therefore, all in all, we obtain that the total digit sum of $X, X+1, \ldots, X+999$ equals

$$(1000-r)\cdot d + r\cdot(d+1) + 13500 = 1000d + r + 13500 = N,$$

as required.

Part 2, by induction. Given a non-negative integer n, denote by s_n the digit sum of n and by $S(n)$ the total digit sum of $n, n+1, \ldots, n+999$, that is,

$$S(n) = s_n + s_{n+1} + \cdots + s_{n+999}.$$

We proceed by induction. As a first step, we show that the 1000 numbers $N \in \{13500, \ldots, 14499\}$ are all contagious. As a second step, we show that if N is contagious, then $N+1000$ is contagious. Combined, this implies that all $N \geq 13500$ are contagious.

For the first step, note that for any integer $n \geq 0$ we have

$$S(n+1) - S(n) = (s_{n+1} + \cdots + s_{n+1000}) - (s_n + \cdots + s_{n+999}) = s_{n+1000} - s_n.$$

Thus, for $0 \leq X \leq 999$, we have $S(X+1) = S(X) + 1$, since the number $X + 1000$ has an extra digit 1 in front of the (up to three-digit) number X. Since $S(0) = 13500$ by Part 1, we get $S(X) = 13500 + X$ for $0 \leq X \leq 999$. Therefore all $N \in \{13500, \ldots, 14499\}$ are indeed contagious.

For the second step, suppose that N is contagious, that is, there exist 1000 consecutive integers $X, X+1, \ldots, X+999$ with total digit sum N. Take any integer i such that $10^i > X + 999$. Then the 1000 consecutive integers

$$10^i + X, 10^i + X + 1, \ldots, 10^i + X + 999$$

have a total digit sum equal to $N + 1000$ (since each number got an extra digit 1 and, possibly, several zeroes).

Part 2, by discrete continuity. We make three observations:

(A) For any integer $n \geq 0$, we have $S(n+1) - S(n) \leq 1$.

- Indeed, as before, we have

  $$S(n+1) - S(n) = (s_{n+1} + \cdots + s_{n+1000}) - (s_n + \cdots + s_{n+999}) = s_{n+1000} - s_n.$$

  Note that the numbers $n+1000$ and n have the same last three digits. We distinguish two cases:

a) If the fourth digit of n from the right is less than 9, then the digits of $n + 1000$ and n differ only in that position and we have $s_{n+1000} - s_n = 1$. (If n is 3-digit, this is true too.)

b) Otherwise, suppose that there are $d \geq 1$ consecutive digits 9 just in front of the last three digits of n. Then $s_{n+1000} - s_n = 1 - 9d < 1$, because the resulting number will have d zeroes in place of the nines, and the digit to the left of the nines increased by one.

(B) We have $S(0) = 13500$.

- By the same argument as in Part 1 we get $S(0) = 3 \cdot 100 \cdot (0 + 1 + \cdots + 9) = 13500$.

(C) The sequence $S(n)$ is unbounded as $n \to \infty$.

- For instance, setting $n = 10^k - 1$ we get $S(n) \geq s_n = 9k$.

It remains to put the observations together. By (B), the number $n = 13500$ is contagious. Now fix $N \geq 13501$. Since the sequence $(S(n))_{n=0}^{\infty}$ is unbounded, there exists an integer $k \geq 1$ such that $S(k) \geq N$. Take the smallest such k. By minimality of k we have $S(k-1) \leq N - 1$. Combining this with (A) we now deduce

$$N \leq S(k) \leq 1 + S(k-1) \leq 1 + (N-1) = N,$$

hence $S(k) = N$ implying that N is contagious.

I-3

Let ABC be an acute scalene triangle with circumcircle ω and incenter I. Suppose the orthocenter H of BIC lies inside ω. Let M be the midpoint of the longer arc BC of ω. Let N be the midpoint of the shorter arc AM of ω.

Prove that there exists a circle tangent to ω at N and tangent to the circumcircles of BHI and CHI.

Denote the circumcircles of BHI and CHI by ω_1 and ω_2 and their centers by O_1 and O_2, respectively. Let O be the center of ω. Let R be the radius of ω.

Since H is the orthocenter of triangle BIC it follows that I is the orthocenter of triangle BHC. Therefore

$$\angle HIB = 180° - (\angle BHI + \angle IBH) = 180° - (90° - \angle CBH + 90° - \angle BHC) = 180° - \angle HCB,$$

and analogously we get $\angle CIH = 180ř - \angle CBH$ and $\angle BIC = 180ř - \angle BHC$.

Denote by r the radius of circle ω_1, then from sine law we get

$$2r = \frac{HB}{\sin\angle HIB} = \frac{HB}{\sin(180° - \angle HIB)} = \frac{HB}{\sin\angle HCB} =$$
$$= \text{diameter of circumcircle of the triangle } BHC.$$

Using the same argument for triangles CIH and BIC we see that r is equal to radii of ω_1, ω_2, circumcircles of BIC and BHC.

From the following angle chase it follows that

$$\angle BHC = 180° - \angle BIC = 180° - \left(180° - \frac{1}{2}\angle CBA - \frac{1}{2}\angle BCA\right) =$$
$$= \frac{1}{2}(\angle CBA + \angle BCA) = 90° - \frac{1}{2}\angle BAC.$$

Since H lies inside ω and $\angle BAC$ is acute we conclude that

$$\angle BAC < \angle BHC = 90° - \frac{1}{2}\angle BAC < 90°$$

so

$$2r = \text{diameter of circumcircle of } BHC = \frac{BC}{\sin\angle BHC} < \frac{BC}{\sin\angle BAC} = 2R,$$

thus $r < R$.

Let $\angle BAC = \alpha$, $\angle CBA = \beta$, $\angle ACB = \gamma$. Then

$$\angle BO_1I = 2\angle BHI = 2(90° - \angle CBH) = 2\angle ICB = \gamma,$$

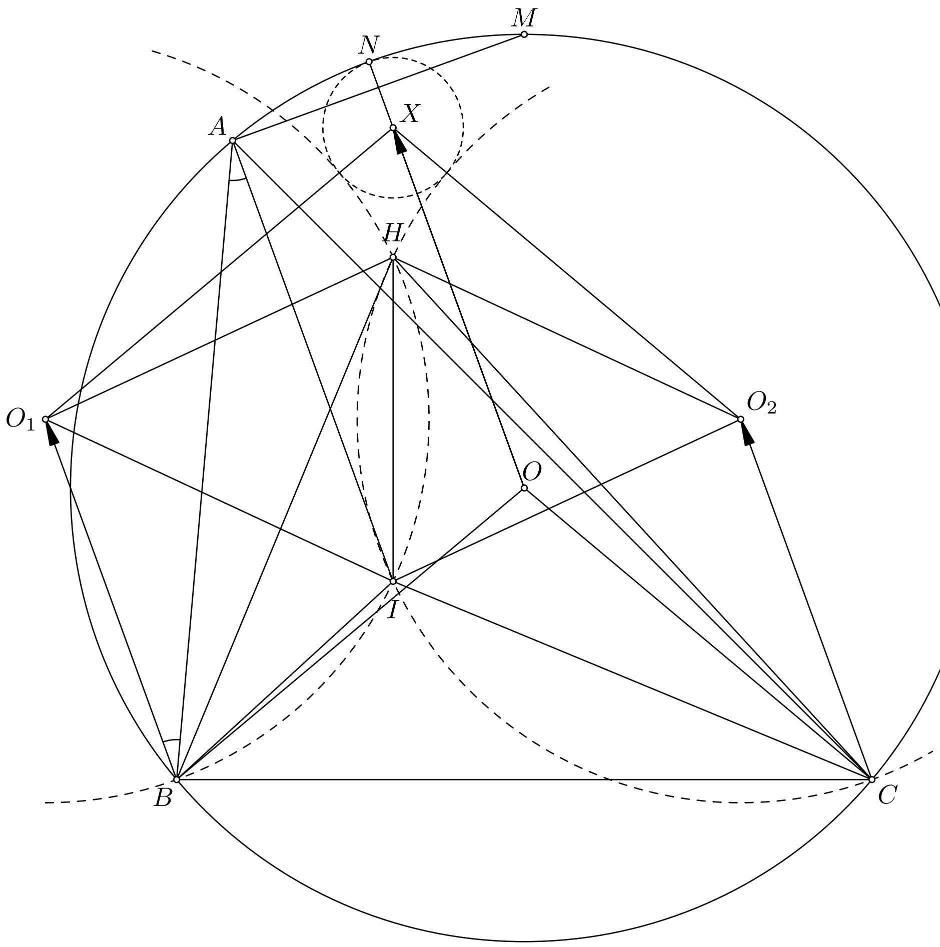
so

$$\angle IBO_1 = 90^\circ - \frac{1}{2}\angle BO_1I = 90^\circ - \frac{\gamma}{2} = \frac{\alpha+\beta}{2},$$

and finally

$$\angle ABO_1 = \angle IBO_1 - \angle IBA = \frac{\alpha+\beta}{2} - \frac{\beta}{2} = \frac{\alpha}{2} = \angle BAI.$$

This shows that $BO_1 \parallel AI$, and moreover, rays $BO_1^{\to}$, $AI^{\to}$ determine opposite directions. Similarly, rays $CO_2^{\to}$, $AI^{\to}$ are parallel and determine opposite directions. Therefore these rays are parallel and $BO_1^{\to}$, $CO_2^{\to}$ determine the same direction. Since $BO_1 = r = CO_2$, it follows that vectors $\overrightarrow{BO_1}$, $\overrightarrow{CO_2}$ are equal. Denote this vector by $\overrightarrow{v}$.



Note that $ON \perp AM$. Moreover

$$\angle IAM = \angle IAC + \angle CAM = \angle IAC + \angle CBM = \\ = \angle IAC + \frac{1}{2}(180^\circ - \angle BMC) = \angle IAC + \frac{1}{2}(180^\circ - \angle BAC) = 90^\circ,$$

so $AM \perp AI$, hence $ON \parallel AI \parallel BO_1 \parallel \overrightarrow{v}$. Let X be a point such that $\overrightarrow{OX} = \overrightarrow{v}$. Since $ON \parallel \overrightarrow{v}$, X lies on line ON. It actually lies on ray $ON^{\to}$ since rays $ON^{\to}$, $AI^{\to}$ determine opposite directions.

Note that translation by $\overrightarrow{v}$ maps triangle BCO to triangle O_1O_2X. Therefore $O_1X = BO = R$ and $O_2X = CO = R$.

Let ω' be the circle centered at X with radius $R - r > 0$.

Observe that $O_1X = R = r + (R - r)$, so ω' is tangent externally to ω_1. For similar reason it is tangent externally to ω_2. Moreover $OX = r = R - (R - r) = ON - XN$, so ω' is tangent to ω internally at point N.

I-4

Find all positive integers n for which there exist positive integers x_1, x_2, …, x_n such that

$$\frac{1}{x_1^2} + \frac{2}{x_2^2} + \frac{4}{x_3^2} + \cdots + \frac{2^{n-1}}{x_n^2} = 1.$$

Answer. Solutions exist for all positive integers n except for $n = 2$.

Solution:

- $n = 1$:

  Here, $x_1 := 1$ provides a solution, since

  $$\frac{1}{1^2} = 1\,.$$

- $n = 2$:

  Here, no solution exists. Indeed, $x_1 = 1$ or $x_2 = 1$ yields $\frac{1}{x_1^2} + \frac{2}{x_2^2} > 1$, while $x_1, x_2 \geq 2$ leads to

  $$\frac{1}{x_1^2} + \frac{2}{x_2^2} \leq \frac{1}{4} + \frac{2}{4} = \frac{3}{4} < 1\,.$$

- $n = 4$:

  Here, $(x_1, x_2, x_3, x_4) := (3, 3, 3, 6)$ provides a solution, since

  $$\frac{1}{3^2} + \frac{2}{3^2} + \frac{4}{3^2} + \frac{8}{6^2} = \frac{7}{9} + \frac{8}{36} = \frac{7}{9} + \frac{2}{9} = 1\,.$$

- Induction step from n to $(n + 2)$:

  Let $(y_1, y_2, \ldots, y_n)$ be a solution for n. I.e.,

  $$\frac{1}{y_1^2} + \frac{2}{y_2^2} + \frac{4}{y_3^2} + \cdots + \frac{2^{n-1}}{y_n^2} = 1\,.$$

  Then

  $$(x_1, x_2, \ldots, x_{n+2}) := (2, 2, 4y_1, 4y_2, \ldots, 4y_n)$$

is a solution for $(n+2)$, since

$$\begin{aligned}\frac{1}{x_1^2}+\frac{2}{x_2^2}+\frac{4}{x_3^2}+\cdots+\frac{2^{n+1}}{x_{n+2}^2} &= \frac{1}{2^2}+\frac{2}{2^2}+\frac{4}{(4y_1)^2}+\cdots+\frac{2^{n+1}}{(4y_n)^2}\\ &= \frac{1}{4}+\frac{2}{4}+\frac{4}{16}\left[\frac{1}{(y_1)^2}+\cdots+\frac{2^{n-1}}{(y_n)^2}\right]\\ &= \frac{3}{4}+\frac{1}{4}\cdot 1\\ &= 1\,.\end{aligned}$$

Remark. There are other induction steps possible. For example from n to $(n+3)$:

Let $(y_1, y_2, \ldots, y_n)$ be a solution for n. I.e.,

$$\frac{1}{y_1^2}+\frac{2}{y_2^2}+\frac{4}{y_3^2}+\cdots+\frac{2^{n-1}}{y_n^2}=1\,.$$

Then

$$(x_1, x_2, \ldots, x_{n+3}) := (3, 3, 3, 6y_1, 6y_2, \ldots, 6y_n)$$

is a solution for $(n+3)$, since

$$\begin{aligned}\frac{1}{x_1^2}+\frac{2}{x_2^2}+\frac{4}{x_3^2}+\cdots+\frac{2^{n+2}}{x_{n+3}^2} &= \frac{1}{3^2}+\frac{2}{3^2}+\frac{4}{3^2}+\frac{8}{(6y_1)^2}+\cdots+\frac{2^{n+2}}{(6y_n)^2}\\ &= \frac{1}{9}+\frac{2}{9}+\frac{4}{9}+\frac{8}{36}\left[\frac{1}{(y_1)^2}+\cdots+\frac{2^{n-1}}{(y_n)^2}\right]\\ &= \frac{7}{9}+\frac{2}{9}\cdot 1\\ &= 1\,.\end{aligned}$$

In order to complete this approach, of course, solutions have to be provided for $n = 1$, 3, and 5.

In fact, every solution $(z_1, \ldots, z_k)$ for k yields an induction step from n to $(n+k-1)$. Indeed, if $(y_1, \ldots, y_n)$ is a solution then

$$(z_1, \ldots, z_{k-1}, z_k \cdot y_1, \ldots, z_k \cdot y_n)$$

is a solution, too. The two constructions presented above belong to $(2, 2, 4)$ and $(3, 3, 3, 6)$.

So it is conceivable that someone finds an induction from n to, say, $(n+6)$. In this case, solutions for six suitable small values of n would be necessary in order to complete the approach.