Problem I–1

Let $\mathbb{N}$ be the set of positive integers. Determine all positive integers k for which there exist functions $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{N}$ such that g assumes infinitely many values and such that

$$f^{g(n)}(n) = f(n) + k$$

holds for every positive integer n.

(*Remark.* Here, f^i denotes the function f applied i times, i.e., $f^i(j) = \underbrace{f(f(\dots f(f}_{i \text{ times}}(j))\dots)).$)

Problem I–2

We call a positive integer N *contagious* if there exist 1000 consecutive non-negative integers such that the sum of all their digits is N. Find all contagious positive integers.

Problem I–3

Let ABC be an acute scalene triangle with circumcircle ω and incenter I. Suppose the orthocenter H of BIC lies inside ω. Let M be the midpoint of the longer arc BC of ω. Let N be the midpoint of the shorter arc AM of ω.

Prove that there exists a circle tangent to ω at N and tangent to the circumcircles of BHI and CHI.

Problem I–4

Find all positive integers n for which there exist positive integers $x_1, x_2, \dots, x_n$ such that

$$\frac{1}{x_1^2} + \frac{2}{x_2^2} + \frac{4}{x_3^2} + \dots + \frac{2^{n-1}}{x_n^2} = 1.$$

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.